Draining Films in Unsteady Withdrawal

JOHN A. TALLMADGE

Drexel University, Philadelphia, Pennsylvania

When a short plate is withdrawn from a liquid bath, a thin film clings to the plate. Van Rossum (6) and Chase and Gutfinger (1) have predicted, when capillary and inertial effects are negligible, that the film thickness at the bath surface in unsteady withdrawal is

$$h_R = \sqrt{\frac{\mu}{\rho g}\frac{L}{t_w}} = \sqrt{\frac{\mu U_w}{\rho g}} \quad \text{or} \quad T_R = 1 \qquad (1)$$

This thickness is not realized precisely at the bath surface because of the presence of the meniscus, but it serves as an estimate and T_R is taken here as a reference quantity. The film profile for this case was predicted to be parabolic, that is, the square root function of (x/L).

$$h = \sqrt{\frac{\mu}{\rho g}\cdot\frac{x}{t_w}} = h_R\sqrt{\frac{x}{L}} \quad \text{or} \quad T = \sqrt{\frac{x}{L}} \qquad (2)$$

These expressions are consistent with the flow equation, $\mu(\partial^2 u/\partial y^2) = \rho g$, and the differential thickness equation of Jeffreys (3):

$$\frac{\partial h}{\partial t} = -\frac{\rho g h^2}{\mu}\frac{\partial h}{\partial x} \qquad (3)$$

The film profile satisfies the boundary condition of $h = 0$ at $x = 0$, the top of the film.

The possibility of a constant thickness region has been suggested in a diagram, Figure 3B of reference 5, and by Jeffreys (3), who noted that a constant thickness film $(\partial h/\partial x = 0)$ satisfies Equation (3). More important, Groenveld (2) experimentally observed such a region at $0.5 < x/L < 0.9$ for one set of conditions. His experimental conditions can be expressed as a withdrawal time of 3.4 sec. (L of 39 cm. and U_w of 11.4 cm./sec.), a N_{Ca} of 4.1 (25 poise and 70 dyne/cm. sugar syrup) and a N_{Fp} of 5.1 (1.38 g./ml.). He found that his constant thickness film was thinner than that predicted by profile Equation (2). The observed thickness agreed within about 10% with a profile line drawn at a T value of 0.66.

Groenveld (2) developed some special case profiles for the short time deviations from Equation (2), but did so only for limited ranges of N_{Ca}. (Groenveld's ranges are discussed at the end of this note. Here N_{Ca} may be considered to be a dimensionless speed of withdrawal.)

GENERAL PROFILE FOR UNSTEADY WITHDRAWAL

The first purpose of this note is to develop expressions which describe unsteady withdrawal profiles over a wide range of N_{Ca}. The best wide-speed theory for continuous withdrawal is the experimentally verified, four-force inertial theory (4):

$$N_{Ca} = 1.09\, D_o^{3/2} + D_o^2 + 0.5\,\beta D_o^2 \qquad (4)$$

where

$$\beta = \exp\left[-5.13\, N_{Fp}^2/D_o\, N_{Ca}^{4/3}\right] = \exp\left[-1.71\, D_o^2/N_{Re}\, N_{Ca}^{4/3}\right] \qquad (4a)$$

The D_o solution to Equation (4) has been converted (footnote in Table 1) to the T_o function of (N_{Ca}, N_{Fp}) or (N_{Ca}, β). Table 1 shows that T_o is a normalized parameter which increases with increasing N_{Ca} and is a weak function of β (that is, N_{Fp} or N_{Re}). We take the T_o value for continuous withdrawal as the constant-region value (T_c) for unsteady withdrawal. Thus using Equation (4) or any theory for continuous withdrawal

$$T_c = T_o\,(N_{Ca}, N_{Fp}) \quad \text{or} \quad T_c = T_o\,(N_{Ca}, N_{Re}) \qquad (5)$$

The parabolic profile holds as long as h is smaller than h_c, or

$$T < T_c, \quad \text{or} \quad h < T_o\sqrt{\frac{\mu U_w}{\rho g}}, \quad \text{or} \quad \frac{x}{L} < T_o^2 \qquad (6)$$

Thus the general profile for unsteady withdrawal is

$$\begin{aligned} &\text{at} \quad \frac{x}{L} < T_o^2, \qquad T = \sqrt{\frac{x}{L}} \\ &\text{at} \quad T_o^2 < \frac{x}{L} < 1, \quad T = T_o \end{aligned} \qquad (7)$$

Equation (7) is a new expression, requiring only the continuous withdrawal values of T_o [such as Equation (4) or Table 1]. For the conditions studied by Groenveld ($N_{Ca} = 4$, $N_{Fp} = 5$), the predicted value of $T_o = 0.71$ agrees within experimental precision with the value used by Groenveld to describe the profile and the $x/L = 0.49$ distance of Equation (6) agrees with the observed location.

Equation (7) is an approximate profile. It was not designed to describe precisely either thickness in the meniscus or slope at the critical point.

SPECIAL CASE PROFILES

Approximate, but wide-speed profiles can be estimated as a function of N_{Ca} only by use of either of the two limiting cases shown in Table 1, namely, $\beta = 0$ (low N_{Re}) or $\beta = 1$ (large N_{Re}).

Limited speed expressions can also be estimated using limited ranges of N_{Ca}. For low $N_{Ca} < 10^{-2}$ the solution is given by

$$\begin{aligned} &\text{at} \quad (x/L) < 0.9\, N_{Ca}^{1/3}, \quad T = (x/L)^{1/2} \\ &\text{at} \quad (x/L) > 0.9\, N_{Ca}^{1/3}, \quad T = 0.94\, N_{Ca}^{1/6} \end{aligned} \qquad (8)$$

For moderate $N_{Ca} \sim 1$, $T_o = 0.62 \pm 0.04$ at $(x/L) > 0.36 \pm 0.06$. For very large N_{Ca} and small N_{Re}, $T_o \rightarrow 1$ and $T_o^2 \rightarrow 1$. For large N_{Ca} and N_{Re}, $T_o \rightarrow 0.82$ and $T_o^2 \rightarrow 0.67$.

Interpolation equations such as $T_o^2 = a + bN_{Ca}$ can be fit over narrow N_{Ca} ranges where desired.

CRITICAL DRAIN TIME AND POSTWITHDRAWAL DRAINAGE PROFILES

The second purpose of this note is to describe the film thickness profiles as they change after withdrawal has stopped. Here the plate is in contact with the bath. It has been shown that the (x/L) junction between the parabolic and constant profile propagates downward with the velocity U_j

$$U_j = \left(\frac{\partial x}{\partial t}\right)_h = \frac{(\partial h/\partial t)_x}{(\partial h/\partial x)_t} = \frac{\rho g h_c^2}{\mu} = \left(\frac{h_c^2 \rho g}{U_w}\right) U_w \qquad (9)$$

Thus

$$U_j = T_o^2 U_w \quad (10)$$

As a result the parabolic profile will travel the distance $(L - x_c)$ in a critical drain time t_d^* of

$$t_d^* = \frac{(L - x_c)}{U_j} = \frac{L}{U_j}\left[1 - \left(\frac{x_c}{L}\right)\right] = \frac{L}{U_w}\left(\frac{1 - T_o^2}{T_o^2}\right) \quad (11)$$

or

$$t_d^* = t_w\left(\frac{1 - T_o^2}{T_o^2}\right) \quad (12)$$

The x/L junction and film profile for the first portion of post withdrawal drainage (up to t_d^*) are given by

$$\left(\frac{x}{L}\right)_j = T_o^2 + \frac{U_j t_d}{L} = T_o^2 + T_o^2\left(\frac{t_d}{t_w}\right) \quad (13)$$

$$\text{at} \quad \frac{x}{L} < \left(\frac{x}{L}\right)_j, \quad h = \sqrt{\frac{x}{L}}\sqrt{\frac{\mu L}{\rho g(t_w + t_d)}}$$

$$\text{at} \quad \frac{x}{L} > \left(\frac{x}{L}\right)_j, \quad h = T_o\sqrt{\frac{\mu U_w}{\rho g}} \quad (14)$$

The shape of the entire film after the critical drain time t_d^* is given by the curved profile in Equation (14). Therefore a constant thickness region will exist for short drain times $t_d < t_d^*$, but the entire film will be parabolic for $t_d > t_d^*$.

The above equations also represent better descriptions of entrained mass and film profiles in the more complex problem of removal (5).

COMPARISON WITH GROENVELD

Time and profile Equations (12) to (14) for postwithdrawal drainage are completely new, but Equations (8) and (9) are given by Groenveld (2). The unsteady withdrawal Equations (5), (6), and (7) and condition Equations (10) and (11) are new in the sense they are general for any T_o, but were based on those developed by Groenveld (2) for special cases. To illustrate the generality of these T_o expressions, we used Equation (4) because it has been verified (4) for a 5½ cycle range of N_{Ca} (that is, from 10^{-4} to 4×10^1). As noted with Equation (5), however, any verified theory for T_o may be used with the equations presented here.

TABLE 1. T_o FUNCTION OF CONTINUOUS WITHDRAWAL EQUATION (4)*

Speed, N_{Ca}	At $\beta = 0$ (Small N_{Re})	At $\beta = 1$ (Large N_{Re})
10^{-4}	0.20	0.20
10^{-3}	0.28	0.28
10^{-2}	0.39	0.38
10^{-1}	0.52	0.48
10^0	0.65	0.58
10^1	0.77	0.67
10^2	0.85	0.72
10^3	0.91	0.76
10^4	0.95	0.78
∞	1.00	0.82

* For a given N_{Ca}, T_o (N_{Ca}, β) of Equation (4) lies between the two extremes shown above. Note that $T_o \equiv D_o/N_{Ca}^{1/2}$ by definition. Here N_{Re} may be calculated using $N_{Re} = D_o^3/3N_{Fp}^2$.

On the other hand, Groenveld's equations were based on special cases for limited ranges in N_{Ca}. One case used the T_o verified for two cycles of N_{Ca} from 10^{-4} to 10^{-2} [Equation (8)]. A second case used a $T_o = 0.66$ verified for one cycle ($N_{Ca} \sim 1$). Other special cases were noted for unverified ranges, but they are controversial at the moment. Discussion of the controversies regarding continuous withdrawal is beyond the scope of this note. We note parenthetically that the untested region of $N_{Ca} > 10^2$ is less likely to occur in unsteady withdrawal than in continuous withdrawal, because of mechanical limitations in withdrawing short objects at high speeds.

SUMMARY

During unsteady withdrawal a region of constant thickness exists at the lower part of the film. The profile shape can be described in terms of a critical distance [x_c/L of Equation (6)], above which the profile is parabolic and below which the profile is flat. The influence of N_{Ca} on profile shape is described here [Equation (7)] for a wide range of N_{Ca}. The secondary influence of N_{Fp} (or N_{Re}) is also described by Equation (7). The profile has been verified experimentally at one set of conditions.

The critical drain time needed to obtain completely parabolic profiles in postwithdrawal drainage is also given for a wide range of N_{Ca} and N_{Fp} [Equation (12)].

Postscript: After this note was written, the above equations were applied to the case of liquid lowering [Tallmadge, J. A., *J. Phys. Chem.* **75**, 383 (1971)].

NOTATION

N_{Ca} = capillary number, $U_w(\mu/\sigma)$
D_o = continuous withdrawal thickness, $h_o(\rho g/\sigma)^{1/2}$
N_{Fp} = fluid property number, $\mu(g/\rho\sigma^3)^{1/4}$
g = gravitational acceleration, cm./sec.^{2}
h = film thickness, cm.
L = distance from top of film to bath surface, cm.
N_{Re} = Reynolds number, $h_o\ (U_w - U_B)\rho/\mu$
N_{Re}' = modified Reynolds number, $h_o U_w \rho/\mu$
t = time, sec.
t_d = postwithdrawal drain time, sec.
t_w = withdrawal time, sec.
T = $h(\rho g/\mu U_w)^{1/2}$
U_B = bulk velocity, cm./sec.
U_j = junction velocity, cm./sec.
U_w = withdrawal velocity, cm./sec.
x = coordinate, distance from top of film, cm.

Greek Letters

β see Equation (4a)
μ = viscosity, poise
ρ = density, g./ml.
σ = surface tension, dyne/cm.

Subscripts

c = at junction between parabolic and constant profiles
o = in continuous withdrawal
R = reference thickness, Equation (1)

LITERATURE CITED

1. Chase, C. A., and Chaim Gutfinger, *AIChE J.*, **13**, 393 (1967).
2. Groenveld, P., *ibid.*, **17**, 489 (1971).
3. Jeffreys, H., *Proc. Cambridge Phil. Soc.*, **26**, 204 (1930).
4. Soroka, J. A., and J. A. Tallmadge, *AIChE J.*, **17**, 504 (1971).
5. Tallmadge, J. A., and Chaim Gutfinger, *Ind. Eng. Chem.*, **59**, (11), 18 (1967); Errata in **60**, (2), 74 (1968).
6. Van Rossum, J. J., *Appl. Sci. Res.*, **A7**, 121 (1958).